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Free convection from a disk rotating in a vertical plane

By S. S. CHAWLA AND A. R. VERMA
Department of Mathematics, Indian Institute of Technology, Kharagpur

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An exact solution of the free convective flow of a viscous incompressible fluid from a heated disk, rotating in a vertical plane, is obtained. The non-axisymmetric fluid motion consists of two parts; the primary von Kármán axisymmetric flow and the secondary buoyancy-induced cross-flow. A highly accurate solution of the energy equation is also derived for its subsequent use in the analysis of the cross-flow.

1. Introduction

The axisymmetry is destroyed in rotating flows when translational velocities are imposed on a basic symmetric flow. A class of such flows has been studied by Rott & Lewellen (1967). A particular case of this class is the steady flow due to a uniform free stream past a rotating disk. All the cases discussed by Rott & Lewellen belong to the general class of exact solutions of the Navier–Stokes equations analysed by Lin (1957). In the present paper we attempt to deal with yet another case of the same class by including the energy equation, coupled with the momentum equations through buoyancy. To this end, we consider the fluid motion and thermal field induced by a heated vertical disk of infinite extent rotating in contact with a viscous incompressible fluid. The symmetry of the basic von Kármán flow is destroyed by the buoyancy-induced cross-flow. A suitable transformation uncouples the governing momentum and energy equations. The solution of the energy equation depends upon the primary axisymmetric flow due to a rotating disk. The secondary cross-flow is governed by the thermal field as well as the primary von Kármán flow. The secondary flow is found to vary on two lengthscales. The thermal forcing tends to dominate the centrifugal action as the Prandtl number of the fluid takes on comparatively larger values.

It is worthwhile to mention that no solution exists in the absence of rotation of the disk.

2. Mathematical formulation

Consider an infinite vertical disk placed at $z = 0$ in contact with a viscous fluid of semi-infinite extent. The disk is rotating with constant angular velocity Ω about the z-axis, which is horizontal. The disk is kept at a constant temperature T_w whereas the temperature of the fluid in the far-off region is T_∞. We take the Cartesian coordinate system (x, y, z) in a non-rotating frame of reference with the x-axis along the upward vertical aligned with the negative direction of gravity g. The equations governing the velocity components (u, v, w), the pressure p, and the temperature T of an incompressible fluid are

$$u_x + v_y + w_z = 0, \tag{2.1}$$

$$uu_x + vu_y + wu_z = -\rho^{-1}p_x + \nu(u_{xx} + u_{yy} + u_{zz}) + g\beta(T - T_\infty), \tag{2.2}$$

$$uv_x + vv_y + wv_z = -\rho^{-1}p_y + \nu(v_{xx} + v_{yy} + v_{zz}), \tag{2.3}$$

$$uw_x + vw_y + ww_z = -\rho^{-1}p_z + \nu(w_{xx} + w_{yy} + w_{zz}), \tag{2.4}$$

$$uT_x + vT_y + wT_z = \kappa(T_{xx} + T_{yy} + T_{zz}), \tag{2.5}$$

wherein we have not taken viscous dissipation into account. ρ, ν, β and κ are respectively the density, kinematic viscosity, coefficient of thermal expansion and thermal conductivity of the fluid. The variation in density is taken into account only in the derivation of the buoyancy force, while other density variations are neglected within the framework of constant-property fluid. The boundary conditions are

$$u = -y\Omega, \quad v = x\Omega, \quad w = 0, \quad T = T_w \quad \text{at} \quad z = 0, \tag{2.6a}$$

$$u \to 0, \quad v \to 0, \quad T \to T_\infty \quad \text{as} \quad z \to \infty. \tag{2.6b}$$

The resulting flow is a member of the general class of exact solutions of the Navier–Stokes equations given by Lin (1957), and consistently with the continuity equation it is appropriate to assume that the velocity, pressure and temperature take the form

$$u = \Omega[-\tfrac{1}{2}xH_\eta - yG] + g\beta(T_w - T_\infty)\,H_1/\Omega, \tag{2.7a}$$

$$v = \Omega[xG - \tfrac{1}{2}yH_\eta] + g\beta(T_w - T_\infty)\,H_2/\Omega, \tag{2.7b}$$

$$w = (\nu\Omega)^{\frac{1}{2}}H, \quad p = -\rho\nu\Omega P, \quad T = T_\infty + \theta(T_w - T_\infty), \tag{2.7c}$$

where H, G, H_1, H_2, P and θ are functions of η defined by

$$\eta = (\Omega/\nu)^{\frac{1}{2}}z. \tag{2.7d}$$

Introducing the expressions (2.7) into the momentum and energy equations (2.2)–(2.5) and equating the coefficients of x, the coefficients of y and terms independent of x and y separately to zero, we arrive at the following set of differential equations:

$$H_{\eta\eta\eta} - HH_{\eta\eta} + \tfrac{1}{2}H_\eta^2 - 2G^2 = 0, \quad G_{\eta\eta} - HG_\eta + GH_\eta = 0, \tag{2.8}$$

$$\theta_{\eta\eta} - \sigma H\theta_\eta = 0, \tag{2.9}$$

$$H_{1\eta\eta} - HH_{1\eta} + \tfrac{1}{2}H_1H_\eta + GH_2 + \theta = 0, \tag{2.10a}$$

$$H_{2\eta\eta} - HH_{2\eta} + \tfrac{1}{2}H_2H_\eta - GH_1 = 0, \tag{2.10b}$$

where $\sigma\,(=\nu/\kappa)$ is the Prandtl number. The boundary conditions (2.6) are transformed into

$$H(0) = 0, \quad H_\eta(0) = 0, \quad G(0) = 1, \quad \theta(0) = 1, \quad H_1(0) = 0 = H_2(0), \tag{2.11a}$$

$$H_\eta(\infty) = 0, \quad G(\infty) = 0, \quad \theta(\infty) = 0, \quad H_2(\infty) = 0, \quad H_1(\infty) = 0. \tag{2.11b}$$

The substitution (2.7) reduces the governing equations to an uncoupled system of differential sets (2.8) (the primary flow), (2.9) (thermal field) and (2.10) (the secondary cross-flow), which can be solved one after the other in that order. The flow field characterized by H and G is the classical von Kármán flow due to the rotating disk. The set of equations (2.8) has been solved by von Kármán (1921), Cochran (1934), Fettis (1955) and Benton (1966). We will make use of Benton's solution in the subsequent analysis. It is obvious that the solution of the coupled differential system (2.10) governing the cross-flow (H_1, H_2) depends upon the solution of the set (2.8) and the energy equation (2.9). None of the available solutions of the energy equation (2.9) can, however, be employed, because they are either known numerically or asymptotically. In §3 we proceed to obtain an analytical–numerical solution of the energy equation.

3. Thermal field

The steady heat transfer from a rotating disk was considered by Millsaps & Pohlhausen (1952), Sparrow & Gregg (1959) and Chao & Greif (1974). They solved the governing equation numerically for various values of the Prandtl number σ. Riley (1964), on the other hand, obtained the rate of heat transfer for large and small values of σ. Asymptotic solutions of the energy equation (2.9) were also obtained by Morgan & Warner (1956), Davies (1959) and Davies & Baxter (1961). Unfortunately none of these solutions serves the purpose under investigation. We have therefore reconsidered the energy equation (2.9) and obtained a highly accurate solution by employing a method based on the ideas of Fettis (1955) and Benton (1966). This is a relatively simple method in which the solution of the differential equation (2.9) is reduced to the solution of a system of linear algebraic equations. Moreover, the solution thus obtained takes into account the effect of curvature of the complete von Kármán flow profile and gives a unified representation to the thermal field for all values of σ. For this purpose we follow Benton (1966) and change to the new variable $\lambda = e^{-c\eta}$, where $c = 0{\cdot}88447$, and write

$$H = c(h(\lambda)-1), \quad G = c^2 g(\lambda), \quad \theta = \lambda^{\sigma-1}K(\lambda), \tag{3.1}$$

so that h, g and K are given by

$$\lambda^3 h''' + \lambda^2(2h'' + hh'' - \tfrac{1}{2}h'^2) + \lambda hh' + 2g^2 = 0, \tag{3.2a}$$

$$\lambda^2 g'' + \lambda(hg' - h'g) = 0, \tag{3.2b}$$

with $$h(1) = 1, \quad h'(1) = 0, \quad c^2 g(1) = 1, \quad h(0) = 0, \quad g(0) = 0, \tag{3.2c}$$

and $$\lambda^2 K'' + (\sigma-1)\lambda K' - (\sigma-1)K + \sigma h[(\sigma-1)K + \lambda K'] = 0, \tag{3.3a}$$

with $$K(1) = 1, \quad K(0) = 0. \tag{3.3b}$$

In the above equations a prime denotes differentiation with respect to λ. We solve (3.2) and (3.3) by substituting a power-series expansion in λ of the form

$$g(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n, \quad h(\lambda) = \sum_{n=1}^{\infty} b_n \lambda^n, \quad K(\lambda) = \sum_{n=1}^{\infty} c_n \lambda^n, \tag{3.4}$$

where a_n and b_n are tabulated in Benton (1966) and c_n is given by the recursion relation

$$(n-1)(n+\sigma-1)c_n = -\sigma \sum_{j=1}^{n-1} (j+\sigma-1)c_j b_{n-j} \quad (n = 1, 2, 3, \ldots). \tag{3.5}$$

These are $n-1$ linear equations in n unknowns c_n. In order to complete the system, we make use of the thermal boundary condition at the disk, namely $K(1) = 1$, to get

$$\sum_{n=1}^{\infty} c_n = 1. \tag{3.6}$$

The boundary condition at $\lambda = 0$ (corresponding to $\eta \to \infty$) is automatically satisfied by the substitution (3.4). The linear system of algebraic equations (3.5), (3.6) can be solved for c_n $(n = 1, 2, 3, \ldots)$ for all values of the Prandtl number σ to any desired order of accuracy. For the purpose of computing c_n (by matrix inversion), a large number of values of b_n were generated from the recursion relation corresponding to (3.2) (see Benton 1966), with

$$b_1 = 2{\cdot}36449, \quad a_1 = 1{\cdot}53678. \tag{3.7}$$

σ	c_1	$-\theta_\eta(0)$	$H_{1\eta}(0)$	$-H_{2\eta}(0)$
0·01	1·0002	0·0087	28·4073	21·1869
0·04	1·0030	0·0333	7·5993	5·2477
0·1	1·0172	0·0766	3·4921	2·1135
0·2	1·0622	0·1361	2·1130	1·0827
0·3	1·1298	0·1849	1·6423	0·7440
0·4	1·2182	0·2263	1·3998	0·5763
0·5	1·3272	0·2623	1·2491	0·4762
0·6	1·4578	0·2943	1·1448	0·4095
0·7	1·6118	0·3231	1·0674	0·3618
0·72	1·6456	0·3286	1·0541	0·3538
0·8	1·7918	0·3495	1·0071	0·3258
0·9	2·0010	0·3737	0·9583	0·2976
1·2	2·8464	0·4371	0·8540	0·2403
1·4	3·6484	0·4734	0·8051	0·2151
1·6	4·7137	0·5080	0·7660	0·1957
1·8	6·1297	0·5370	0·7338	0·1804
2·0	8·0136	0·5620	0·7066	0·1678
2·2	10·5172	0·6042	0·6831	0·1573
2·5	15·9350	0·6280	0·6530	0·1444
3·0	32·2985	0·6826	0·6132	0·1279
5·0	607·4207	0·6920	0·5173	0·0923

TABLE 1

This helped in achieving the absolute accuracy of the present solution up to five significant places. The computed values of c_1 and the corresponding coefficient of heat transfer $\theta_\eta(0)$ are presented in table 1 for some representative values of the Prandtl number. These values of c_1 can be used to generate all other coefficients c_n in (3.4) with the help of the recursion relation (3.5).

4. Free convection

In order to obtain the buoyancy-induced cross-flow, which is governed by (2.10) and (2.11) and evidently depends on all three velocity components of the von Kármán flow, we set

$$c^2H_1 = \gamma_1\lambda h' - 2\gamma_2 g - \frac{2\alpha_1}{\sigma-1}(g-\lambda^{\sigma-1}g) + \frac{\alpha_2\lambda}{\sigma-1}(h'-\lambda^{\sigma-1}h') + \lambda^\sigma h_1(\lambda), \tag{4.1}$$

$$c^2H_2 = 2\gamma_1 g + \gamma_2\lambda h' + \frac{\alpha_1\lambda}{\sigma-1}(h'-\lambda^{\sigma-1}h') + \frac{2\alpha_2}{\sigma-1}(g-\lambda^{\sigma-1}g) + \lambda^\sigma h_2(\lambda), \tag{4.2}$$

where α_1, α_2, γ_1 and γ_2 are constants to be determined. The specific advantage of the above transformation is that it enables us to generate the solution of h_1 and h_2 without having to solve for the system of linear equations into which the resulting differential set is ultimately reduced. Substituting (4.1) and (4.2) into (2.10) and (2.11), we get

$$\lambda[\lambda^2 h_1'' + \lambda(2\sigma h_1' + hh_1' - \tfrac{1}{2}h'h_1) + \sigma(\sigma-1)h_1 + \sigma hh_1 + gh_2] + 2\alpha_1(2\lambda g' + (\sigma-2)g + gh) - \alpha_2(2\lambda^2 h'' + \sigma\lambda h' + \lambda hh') + K = 0, \tag{4.3}$$

$$\lambda[\lambda^2 h_2'' + \lambda(2\sigma h_2' + hh_2' - \tfrac{1}{2}h'h_2) + \sigma(\sigma-1)h_2 + \sigma hh_2 - gh_1] - 2\alpha_2(2\lambda g' + (\sigma-2)g + gh) - \alpha_1(2\lambda^2 h'' + \sigma\lambda h' + \lambda hh') = 0, \tag{4.4}$$

with $$h_1(1) = \frac{2\gamma_2}{c^2}, \quad h_2(1) = \frac{-2\gamma_1}{c^2}, \quad h_1(0) = 0 = h_2(0). \tag{4.5}$$

We now write $h_1(\lambda)$ and $h_2(\lambda)$ as power series in λ of the form

$$h_1(\lambda) = \sum_{n=1}^{\infty} d_n \lambda^n, \quad h_2(\lambda) = \sum_{n=1}^{\infty} e_n \lambda^n, \tag{4.6}$$

where d_n and e_n are given by the recursion relations (after using the expression (3.4) for g, h and K)

$$\begin{aligned}&[(n-1)(n+2\sigma-2)+\sigma(\sigma-1)]\,d_{n-1}+c_n+(2n-2+\sigma)(2\alpha_1 a_n - n\alpha_2 b_n)\\ &\quad = \sum_{j=1}^{n-1}([(n-j)\,\alpha_2 b_{n-j}-2\alpha_1 a_{n-j}]\,b_j - a_{j-1}e_{n-j}-\tfrac{1}{2}(2n-3j+1+2\sigma)\,b_{j-1}d_{n-j}),\end{aligned} \tag{4.7}$$

$$\begin{aligned}&[(n-1)(n+2\sigma-2)+\sigma(\sigma-1)]\,e_{n-1}-(2n-2+\sigma)(n\alpha_1 b_n+2\alpha_2 a_n)\\ &\quad = -\sum_{j=1}^{n-1}([(n-j)\,\alpha_1 b_{n-j}+2\alpha_2 a_{n-j}]\,b_j + a_{j-1}d_{n-j}-\tfrac{1}{2}(2n-3j+1+2\sigma)\,b_{j-1}e_{n-j}),\end{aligned} \tag{4.8}$$

with $n = 1, 2, \ldots$. For $n = 1$, the above recursion relations immediately yield the values of α_1 and α_2 as

$$\frac{-\alpha_1}{2a_1} = \frac{\alpha_2}{b_1} = \frac{c_1}{\sigma(b_1^2+4a_1^2)}, \tag{4.9}$$

where a_1 and b_1 are given by (3.7) and c_1 is given in table 1 for various values of σ. Once α_1 and α_2 are evaluated, all the coefficients d_n and e_n can be obtained from the relations (4.7) and (4.8). Finally, the values of the constants γ_1 and γ_2 are derived from the boundary conditions (4.5) as

$$\gamma_1 = -\tfrac{1}{2}c^2 \sum_{n=1}^{\infty} e_n, \quad \gamma_2 = \tfrac{1}{2}c^2 \sum_{n=1}^{\infty} d_n. \tag{4.10}$$

These determine the solution of the differential set (2.10) and (2.11) completely for all values of the Prandtl number σ. Values of (d_1, e_1) and (γ_1, γ_2) obtained from (4.7), (4.8) and (4.10) respectively, correct to four decimal places, are given in table 2. The coefficients of skin friction $H_{1\eta}(0)$ and $H_{2\eta}(0)$ corresponding to the cross-flow are reproduced in table 1 for selected values of the Prandtl number. The variation of the stress components for the whole range of the Prandtl number is represented in figure 1. The flow functions $H_1(\eta)$ and $H_2(\eta)$, giving the induced cross-flow, are exhibited in figure 2 for various values of the Prandtl number σ.

We note from figure 2 that the profiles for the buoyancy-induced cross-flow are much thicker than the primary von Kármán boundary-layer profile (see Benton 1966) as σ varies through comparatively small values. As σ increases, the thickness of the superposed free-convection boundary layer decreases. The thermal forcing superposes a fluid motion along the x-axis. The centrifugal action of the primary flow tends to divert fluid flow in the negative y-direction. The von Kármán axial velocity directed towards the rotating disk (and proportional to $H(\infty)$) causes convective transport of the induced vorticity towards the surface of the disk. As such the cross-flow varies on two lengthscales. As σ increases, the thickness of the thermal boundary layer decreases (see Millsaps & Pohlhausen 1952), and the effect of the thermal forcing tends to be confined to the neighbourhood of the disk. In these circumstances, the convection imparted by the axial inflow is weakened, as H itself is weaker in the vicinity of the rotating disk. Within a radius $g\beta(T_w - T_\infty)/\Omega^2$, the secondary cross-flow actually dominates the primary von Kármán flow.

σ	e_1	d_1	$-\gamma_1$	γ_2
0·01	15294·5938	11649·0977	4924·3516	−4·2868
0·04	944·7991	698·4194	295·2085	−1·1674
0·1	149·2110	104·0547	44·1794	−0·5114
0·2	37·4084	23·8947	10·3001	−0·2571
0·3	17·0664	10·1784	4·4525	−0·1514
0·4	10·0292	5·7315	2·5176	−0·0871
0·5	6·7986	3·8354	1·6621	−0·0409
0·6	5·0561	2·9023	1·2149	−0·0045
0·7	4·0143	2·4100	0·9545	0·0261
0·72	3·8574	2·3434	0·9158	0·0318
0·8	3·3464	2·1494	0·7911	0·0532
0·9	2·8973	2·0250	0·6832	0·0780
1·2	2·2092	2·0936	0·5219	0·1466
1·4	2·0263	2·3718	0·4803	0·1922
1·6	1·9590	2·8055	0·4655	0·2407
1·8	1·9729	3·4075	0·4691	0·2941
2·0	2·0525	4·2107	0·4875	0·3547
2·2	2·1915	5·2624	0·5189	0·4244
2·5	2·5188	7·4741	0·5924	0·5527
3·0	3·4469	13·7890	0·7984	0·8584
5·0	21·7810	192·2966	4·7595	6·0774

TABLE 2

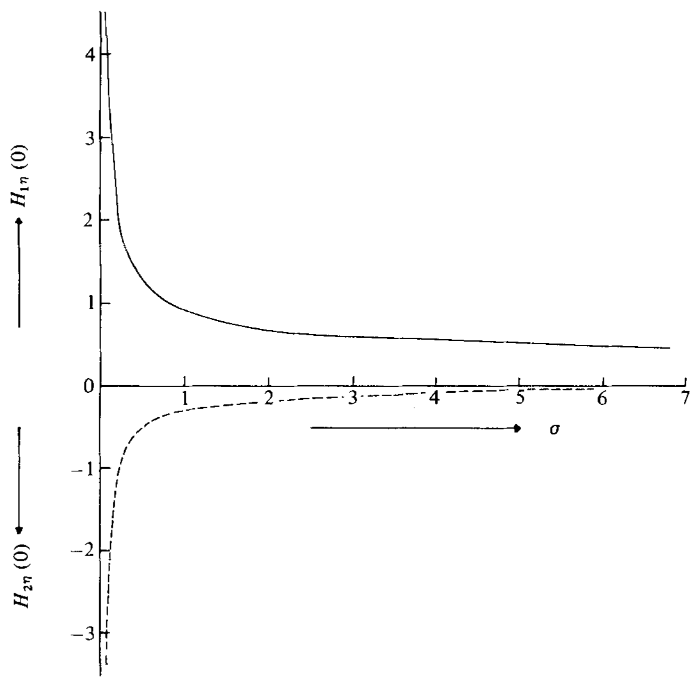

FIGURE 1. Variation of the stress components $H_{1\eta}(0)$ (——) and $H_{2\eta}(0)$ (– – – – –) versus σ.

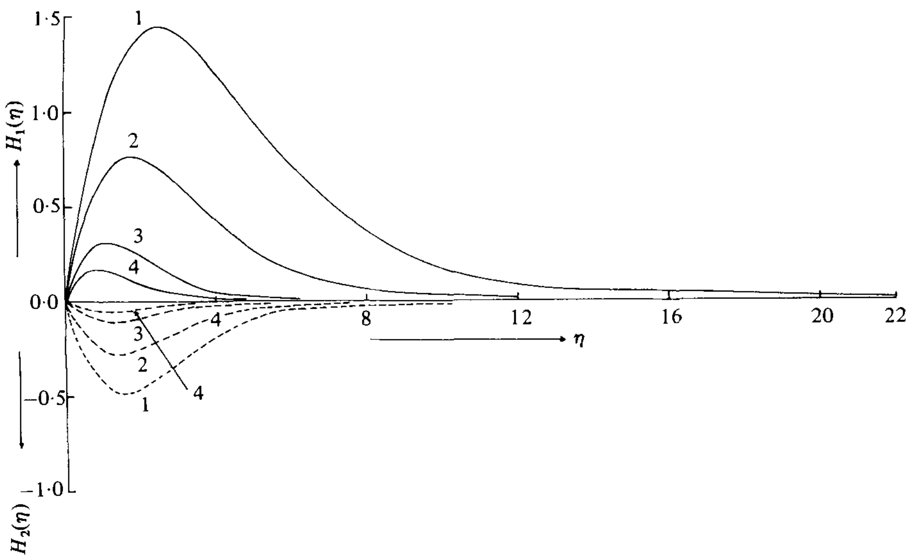

FIGURE 2. Variation of the cross-flow functions $H_1(\eta)$ (——) and $H_2(\eta)$ (- - - - - -) with η for (1) $\sigma = 0{\cdot}4$; (2) 0·72; (3) 2; (4) 5.

On a disk of radius R, the torque associated with the von Kármán flow is $\frac{1}{2}\rho\pi R^4(\nu\Omega^3)^{\frac{1}{2}}\,G_\eta(0)$. For a counter-clockwise rotation of the disk, the coefficient of shear associated with the cross-flow has components $H_{1\eta}(0)$ and $H_{2\eta}(0)$ in the positive x-direction and negative y-direction respectively. Both of these components decrease as σ increases. On a finite disk of radius R (neglecting the edge effects), the resultant force is

$$\rho g\beta\pi R^2(T_{\mathrm{w}}-T_\infty)\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}[H^2_{1\eta}(0)+H^2_{2\eta}(0)]^{\frac{1}{2}}$$

acting through the centre at an angle

$$\phi = \arctan\frac{H_{2\eta}(0)}{H_{1\eta}(0)}$$

against the direction of rotation measured from the vertical x-axis. ϕ varies from $\frac{1}{4}\pi$ to zero as σ takes on values from zero to infinity, implying that the thermal forcing tends to dominate the centrifugal action at large values of the Prandtl number σ.

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